Irreducible representations of knot groups into $\mathrm{SL}(n, \mathbf{C})$

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Abstract

The aim of this article is to study the existence of certain reducible, metabelian representations of knot groups into $\mathrm{SL}(n, \mathbf{C})$ which generalise the representations studied previously by G. Burde and G. de Rham. Under specific hypotheses we prove the existence of irreducible deformations of such representations of knot groups into $\mathrm{SL}(n, \mathbf{C})$.

MSC: 57M25; 57M05; 57M27
Keywords: knot group; Alexander module; variety of representations; character variety.

1 Introduction

In [3], the authors studied the deformations of certain metabelian, reducible representations of knot groups into $\mathrm{SL}(3, \mathbf{C})$. In this paper we continue this study by generalizing all of the results of [3] to the group $\mathrm{SL}(n, \mathbf{C})$ (see Theorem 1.1).

Let Γ be a finitely generated group. The set $R_n(\Gamma) := R(\Gamma, \mathrm{SL}(n, \mathbf{C}))$ of homomorphisms of Γ in $\mathrm{SL}(n, \mathbf{C})$ is called the $\mathrm{SL}(n, \mathbf{C})$-representation variety of Γ. It is a (not necessarily irreducible) algebraic variety. A representation $\rho\colon \Gamma \to \mathrm{SL}(n, \mathbf{C})$ is called *abelian* (resp. *metabelian*) if the restriction of ρ to the first (resp. second) commutator subgroup of Γ is trivial. The representation $\rho\colon \Gamma \to \mathrm{SL}(n)$ is called *reducible* if there exists a proper subspace $V \subset \mathbf{C}^n$ such that $\rho(\Gamma)$ preserves V. Otherwise ρ is called *irreducible.*

Let Γ denote the *knot group* of the knot $K \subset S^3$ i.e. Γ is the fundamental group of the knot complement of K in S^3. Since the ring of complex Laurent polynomials $\mathbf{C}[t^{\pm 1}]$ is a principal ideal domain, the complex *Alexander module* $M(t)$ of K decomposes into a direct sum of cyclic modules. A generator of the order ideal of $M(t)$ is called the *Alexander polynomial* of K. It will be denoted by $\Delta_K(t) \in \mathbf{C}[t^{\pm 1}]$, and it is unique up to multiplication by a

unit $c\,t^k \in \mathbf{C}[t^{\pm 1}]$, $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$. For a given root $\alpha \in \mathbf{C}^*$ of $\Delta_K(t)$ we let τ_α denote the $(t-\alpha)$-torsion of the Alexander module. (For details see Section 2.)

The main result of this article is the following theorem which generalizes the results of [3] where the case $n = 3$ was investigated. It also applies in the case $n = 2$ which was studied in [2] and [12, Theorem 1.1].

1.1 Theorem *Let K be a knot in the 3-sphere S^3. If the $(t-\alpha)$-torsion τ_α of the Alexander module is cyclic of the form $\mathbf{C}[t^{\pm 1}]/(t-\alpha)^{n-1}$, $n \geq 2$, then for each $\lambda \in \mathbf{C}^*$ such that $\lambda^n = \alpha$ there exists a certain reducible metabelian representation ϱ_λ of the knot group Γ into $\mathrm{SL}(n,\mathbf{C})$. Moreover, the representation ϱ_λ is a smooth point of the representation variety $R_n(\Gamma)$, it is contained in a unique (n^2+n-2)-dimensional component R_{ϱ_λ} of $R_n(\Gamma)$. Moreover, R_{ϱ_λ} contains irreducible non-metabelian representations which deform ϱ_λ.*

This paper is organised as follows. In Section 2 we introduce some notations and recall some facts which will be used in this article. In Section 3 we study the existence of certain reducible representations. These representations were previously studied in [13], and we treat the existence results from a more general point of view. Section 4 is devoted to the proof of Proposition 4.1, and it contains all necessary cohomological calculations. In the last section we prove that there are irreducible non-metabelian deformations of the initial reducible representation.

Acknowledgements. Both authors are pleased to acknowledge the support by the French-Tunisian CMCU project n° 12G/1502. Moreover, the first author was supported by the project Erasmus mundus E-GOV-TN. The second author acknowledges support from the ANR projects SGT and ModGroup.

2 Notations and facts

To shorten notation we will simply write $\mathrm{SL}(n)$ (respectively $\mathrm{GL}(n)$) instead of $\mathrm{SL}(n,\mathbf{C})$ (respectively $\mathrm{GL}(n,\mathbf{C})$) and $\mathfrak{sl}(n)$ (respectively $\mathfrak{gl}(n)$) instead of $\mathfrak{sl}(n,\mathbf{C})$ (respectively $\mathfrak{gl}(n,\mathbf{C})$).

Group cohomology. The general reference for group cohomology is K. Brown's book [5]. Let A be a Γ-module. We denote by $C^*(\Gamma; A)$ the

cochain complex, the coboundary operator $\delta\colon C^n(\Gamma;A) \to C^{n+1}(\Gamma;A)$ is given by:

$$\begin{aligned}\delta f(\gamma_1,\ldots,\gamma_{n+1}) &= \gamma_1\cdot f(\gamma_2,\ldots,\gamma_{n+1})\\ &\quad+\sum_{i=1}^{n}(-1)^i f(\gamma_1,\ldots,\gamma_{i-1},\gamma_i\gamma_{i+1},\ldots,\gamma_{n+1})+(-1)^{n+1}f(\gamma_1,\ldots,\gamma_n)\,.\end{aligned}$$

The coboundaries (respectively cocycles, cohomology) of Γ with coefficients in A are denoted by $B^*(\Gamma;A)$ (respectively $Z^*(\Gamma;A)$, $H^*(\Gamma;A)$). In what follows 1-cocycles and 1-coboundaries will be also called *derivations* and *principal derivations* respectively.

Let A_1, A_2 and A_3 be Γ-modules. The cup product of two cochains $u\in C^p(\Gamma;A_1)$ and $v\in C^q(\Gamma;A_2)$ is the cochain $u\smile v\in C^{p+q}(\Gamma;A_1\otimes A_2)$ defined by

$$u\smile v(\gamma_1,\ldots,\gamma_{p+q}) := u(\gamma_1,\ldots,\gamma_p)\otimes\gamma_1\ldots\gamma_p\circ v(\gamma_{p+1},\ldots,\gamma_{p+q})\,. \qquad (1)$$

Here $A_1\otimes A_2$ is a Γ-module via the diagonal action. It is possible to combine the cup product with any Γ-invariant bilinear map $A_1\otimes A_2\to A_3$. We are mainly interested in the product map $\mathbf{C}\otimes\mathbf{C}\to\mathbf{C}$.

2.1 Remark Notice that our definition of the cup product (1) differs from the convention used in [5, V.3] by the sign $(-1)^{pq}$. Hence with the definition (1) the following formula holds:

$$\delta(u\smile v) = (-1)^q\,\delta u\smile v + u\smile\delta v\,.$$

A short exact sequence

$$0\to A_1\xrightarrow{\ i\ }A_2\xrightarrow{\ p\ }A_3\to 0$$

of Γ-modules gives rise to a short exact sequence of cochain complexes:

$$0\to C^*(\Gamma;A_1)\xrightarrow{\ i^*\ }C^*(\Gamma;A_2)\xrightarrow{\ p^*\ }C^*(\Gamma;A_3)\to 0\,.$$

We will make use of the corresponding long exact cohomology sequence (see [5, III. Prop. 6.1]):

$$0\to H^0(\Gamma;A_1)\longrightarrow H^0(\Gamma;A_2)\longrightarrow H^0(\Gamma;A_3)\xrightarrow{\ \beta^0\ }H^1(\Gamma;A_1)\longrightarrow\cdots$$

Recall that the Bockstein homomorphism $\beta^n\colon H^n(\Gamma;A_3)\to H^{n+1}(\Gamma;A_1)$ is determined by the snake lemma: if $z\in Z^n(\Gamma;A_3)$ is a cocycle and if

$\tilde{z} \in (p^*)^{-1}(z) \subset C^n(\Gamma; A_2)$ is any lift of z then $\delta_2(\tilde{z}) \in \mathrm{Im}(i^*)$ where δ_2 the coboundary operator of $C^*(\Gamma; A_2)$. It follows that any cochain $z' \in C^{n+1}(\Gamma; A_3)$ such that $i^*(z') = \delta_2(\tilde{z})$ is a cocycle and that its cohomology class does only depend on the cohomology class represented by z. The cocycle z' represents the image of the cohomology class represented by z under β^n.

2.2 Remark By abuse of notation and if no confusion can arise, we will write sometimes $\beta^n(z)$ for a cocycle $z \in Z^n(\Gamma; A_3)$ even if the map β^n is only well defined on cohomology classes. This will simplify the notations.

The Alexander module Given a knot $K \subset S^3$, we let $X = \overline{S^3 \backslash V(K)}$ denote its complement where $V(K)$ is a tubular neighborhood of K. Let $\Gamma = \pi_1(X)$ denote the fundamental group of X and $h\colon \Gamma \to \mathbf{Z}$, $h(\gamma) = \mathrm{lk}(\gamma, K)$, the canonical projection. Recall also that a knot complement X is aspherical (see [7, 3.F]). In what follows we will identify the cohomology of the knot complement and of the knot group Γ.

Note that there is a short exact splitting sequence

$$1 \to \Gamma' \to \Gamma \to \langle t \mid - \rangle \to 1$$

where $\Gamma' = [\Gamma, \Gamma]$ denote the commutator subgroup of Γ and where the surjection is given by $\gamma \mapsto t^{h(\gamma)}$. Hence Γ is isomorphic to the semi-direct product $\Gamma' \rtimes \mathbf{Z}$. Note that Γ' is the fundamental group of the infinite cyclic covering X_∞ of X. The abelian group $\Gamma'/\Gamma'' \cong H_1(X_\infty, \mathbf{Z})$ turns into a $\mathbf{Z}[t^{\pm1}]$-module via the action of the group of covering transformations which is isomorphic to $\langle t \mid - \rangle$. The $\mathbf{Z}[t^{\pm1}]$-module $H_1(X_\infty, \mathbf{Z})$ is a finitely generated torsion module called the *Alexander module* of K. Note that there are isomorphisms of $\mathbf{Z}[t^{\pm1}]$-modules

$$H_*(\Gamma; \mathbf{Z}[t^{\pm1}]) \cong H_*(X; \mathbf{Z}[t^{\pm1}]) \cong H_*(X_\infty, \mathbf{Z})$$

where Γ acts on $\mathbf{Z}[t^{\pm1}]$ via $\gamma\, p(t) = t^{h(\gamma)}\, p(t)$ for all $\gamma \in \Gamma$ and $p(t) \in \mathbf{Z}[t^{\pm1}]$. (See [8, Chapter 5] for more details.) In what follows we are mainly interested in the complex version $\mathbf{C} \otimes \Gamma'/\Gamma'' \cong H_1(\Gamma; \mathbf{C}[t^{\pm1}])$ of the Alexander module. As $\mathbf{C}[t^{\pm1}]$ is a principal ideal domain, the Alexander module $H_1(\Gamma; \mathbf{C}[t^{\pm1}])$ decomposes into a direct sum of cyclic modules of the form $\mathbf{C}[t^{\pm1}]/(t-\alpha)^k$, $\alpha \in \mathbf{C}^* \setminus \{1\}$ i.e. there exist $\alpha_1, \dots \alpha_s \in \mathbf{C}^*$ such that

$$H_1(\Gamma; \mathbf{C}[t^{\pm1}]) \cong \tau_{\alpha_1} \oplus \cdots \oplus \tau_{\alpha_s} \text{ where } \tau_{\alpha_j} = \bigoplus_{i_j=1}^{n_{\alpha_j}} \mathbf{C}[t^{\pm1}]/(t-\alpha_j)^{r_{i_j}}$$

denotes the $(t-\alpha_j)$-torsion of $H_1(\Gamma;\mathbf{C}[t^{\pm1}])$. A generator of the order ideal of $H_1(X_\infty,\mathbf{C})$ is called the *Alexander polynomial* $\Delta_K(t)\in\mathbf{C}[t^{\pm1}]$ of K i.e. $\Delta_K(t)$ is the product

$$\Delta_K(t)=\prod_{j=1}^{s}\prod_{i_j=1}^{n_{\alpha_j}}(t-\alpha_j)^{r_{j_i}}\,.$$

Notice that the Alexander polynomial is symmetric and is well defined up to multiplication by a unit $c\,t^k$ of $\mathbf{C}[t^{\pm1}]$, $c\in\mathbf{C}^*$, $k\in\mathbf{Z}$. Moreover, $\Delta_K(1)=\pm1\neq0$ (see [7]), and hence the $(t-1)$-torsion of the Alexander module is trivial.

For completeness we will state the next lemma which shows that the cohomology groups $H^*(\Gamma;\mathbf{C}[t^{\pm1}]/(t-\alpha)^k)$ are determined by the Alexander module $H_1(\Gamma;\mathbf{C}[t^{\pm1}])$. Recall that the action of Γ on $\mathbf{C}[t^{\pm1}]/(t-\alpha)^k$ is induced by $\gamma\,p(t)=t^{h(\gamma)}p(t)$.

2.3 Lemma *Let $K\subset S^3$ be a knot and Γ its group. Let $\alpha\in\mathbf{C}^*$ and let $\tau_\alpha=\bigoplus_{i=1}^{n_\alpha}\mathbf{C}[t^{\pm1}]/(t-\alpha)^{r_i}$ denote the $(t-\alpha)$-torsion of the Alexander module $H_1(\Gamma;\mathbf{C}[t^{\pm1}])$. Then if $\alpha=1$ we have that τ_1 is trivial and*

$$H^q(\Gamma;\mathbf{C}[t^{\pm1}]/(t-1)^k)\cong\begin{cases}\mathbf{C} & \text{for } q=0,1\\ 0 & \text{for } q\geq 2.\end{cases}$$

Moreover, for $\alpha\neq1$ we have:

$$H^q(\Gamma;\mathbf{C}[t^{\pm1}]/(t-\alpha)^k)\cong\begin{cases}0 & \text{for } q=0 \text{ and } q\geq3,\\ \bigoplus_{i=1}^{n_\alpha}\mathbf{C}[t^{\pm1}]/(t-\alpha)^{\min(k,r_i)} & \text{for } q=1,2.\end{cases}$$

In particular, $H^1(\Gamma;\mathbf{C}[t^{\pm1}]/(t-\alpha)^k)\neq0$ if and only $H_1(\Gamma;\mathbf{C}[t^{\pm1}])$ has non-trivial $(t-\alpha)$-torsion i.e if $\Delta_K(\alpha)=0$.

Proof. Let M be a $\mathbf{C}[t^{\pm1}]$-module, then by the extension of scalars [5, III.3] we have an isomorphism

$$H^q(\Gamma;M)\cong H^q(\mathrm{Hom}_{\mathbf{C}[t^{\pm1}]}(C_*(X_\infty,\mathbf{C}),M).$$

Since $\mathbf{C}[t^{\pm1}]$ is a principal ideal domain, we can apply the universal coefficient theorem and obtain

$$H^q(\Gamma;M)\cong\mathrm{Ext}^1_{\mathbf{C}[t^{\pm1}]}(H_{q-1}(X_\infty,\mathbf{C}),M)\oplus\mathrm{Hom}_{\mathbf{C}[t^{\pm1}]}(H_q(X_\infty,\mathbf{C}),M).$$

Now $H_0(X_\infty,\mathbf{C})\cong\mathbf{C}\cong\mathbf{C}[t^{\pm1}]/(t-1)$ and $H_k(X_\infty,\mathbf{C})=0$ for $k\geq2$ (see [7, Prop. 8.16]) so we can apply the above isomorphisms to the modules $\mathbf{C}[t^{\pm1}]/(t-\alpha)^k$ with $\alpha=1$ or $\alpha\neq1$. Notice also that for $\lambda\neq\alpha$ the multiplication by $(t-\lambda)$ induces an isomorphism of $\mathbf{C}[t^{\pm1}]/(t-\alpha)^k$. $\square$

Representation variety. Let Γ be a finitely generated group. The set of all homomorphisms of Γ into $\mathrm{SL}(n)$ has the structure of an affine algebraic set (see [14] for details). In what follows this affine algebraic set will be denoted by $R(\Gamma, \mathrm{SL}(n))$ or simply by $R_n(\Gamma)$. Let $\rho\colon \Gamma \to \mathrm{SL}(n)$ be a representation. The Lie algebra $\mathfrak{sl}(n)$ turns into a Γ-module via $\mathrm{Ad}\circ\rho$. This module will be simply denoted by $\mathfrak{sl}(n)_\rho$. A 1-*cocycle* or *derivation* $d \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is a map $d\colon \Gamma \to \mathfrak{sl}(n)$ satisfying

$$d(\gamma_1\gamma_2) = d(\gamma_1) + \mathrm{Ad}\circ\rho(\gamma_1)(d(\gamma_2)) \quad , \ \forall\ \gamma_1,\ \gamma_2 \in \Gamma\,.$$

It was observed by André Weil [15] that there is a natural inclusion of the Zariski tangent space $T^{Zar}_\rho(R_n(\Gamma)) \hookrightarrow Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$. Informally speaking, given a smooth curve ρ_ϵ of representations through $\rho_0 = \rho$ one gets a 1-cocycle $d\colon \Gamma \to \mathfrak{sl}(n)$ by defining

$$d(\gamma) := \left.\frac{d\,\rho_\epsilon(\gamma)}{d\,\epsilon}\right|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma\,.$$

It is easy to see that the tangent space to the orbit by conjugation corresponds to the space of 1-coboundaries $B^1(\Gamma; \mathfrak{sl}(n)_\rho)$. Here, $b\colon \Gamma \to \mathfrak{sl}(n)$ is a coboundary if there exists $x \in \mathfrak{sl}(n)$ such that $b(\gamma) = \mathrm{Ad}\circ\rho(\gamma)(x) - x$. A detailed account can be found in [14].

For the convenience of the reader, we state the following result which is implicitly contained in [3, 12, 11]. A detailed proof of the following streamlined version can be found in [10]:

2.4 Proposition *Let M be an orientable, irreducible 3-manifold with infinite fundamental group $\pi_1(M)$ and incompressible tours boundary, and let $\rho\colon \pi_1(M) \to \mathrm{SL}(n)$ be a representation.*

If $\dim H^1(M; \mathfrak{sl}(n)_\rho) = n - 1$ then ρ is a smooth point of the $\mathrm{SL}(n)$-representation variety $R_n(\pi_1(M))$. More precisely, ρ is contained in a unique component of dimension $n^2 + n - 2 - \dim H^0(\pi_1(M); \mathfrak{sl}(n)_\rho)$.

3 Reducible metabelian representations

Recall that every nonzero complex number $\alpha \in \mathbf{C}^*$ determines an action of a knot group Γ on the complex numbers given by $\gamma\, x = \alpha^{h(\gamma)}x$ for $\gamma \in \Gamma$ and $x \in \mathbf{C}$. The resulting Γ-module will be denoted by $\mathbf{C}_\alpha$. Notice that $\mathbf{C}_\alpha$ is isomorphic to $\mathbf{C}[t^{\pm1}]/(t-\alpha)$.

It is easy to see that a map $\Gamma \to \mathrm{GL}(2, \mathbf{C})$ given by

$$\gamma \mapsto \begin{pmatrix} 1 & z_1(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{h(\gamma)} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{h(\gamma)} & z_1(\gamma) \\ 0 & 1 \end{pmatrix} \tag{2}$$

is a representation if and only if the map $z_1\colon \Gamma \to \mathbf{C}_\alpha$ is a derivation i.e.

$$\delta z_1(\gamma_1,\gamma_2) = \alpha^{h(\gamma_1)} z_1(\gamma_2) - z_1(\gamma_1\gamma_2) + z_1(\gamma_1) = 0 \text{ for all } \gamma_1,\gamma_2 \in \Gamma.$$

The representation given by (2) is non-abelian if and only if $\alpha \neq 1$ and the cocycle z is not a coboundary. Hence it follows from Lemma 2.3 that such a reducible non abelian representation exists if and only if α is a root of the Alexander polynomial. These representations were first studied independently by G. Burde [6] and G. de Rham [9].

We extend these considerations to a map $\Gamma \to \mathrm{GL}(3,\mathbf{C})$. It follows easily that

$$\gamma \mapsto \begin{pmatrix} \alpha^{h(\gamma)} & z_1(\gamma) & z_2(\gamma) \\ 0 & 1 & h(\gamma) \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

is a representation if and only if $\delta z_1 = 0$ and $\delta z_2 + z_1 \smile h = 0$ i.e.

$$\begin{cases} \delta z_1(\gamma_1,\gamma_2) = 0 & \text{for all } \gamma_1,\gamma_2 \in \Gamma, \\ \delta z_2(\gamma_1,\gamma_2) + z_1(\gamma_1)h(\gamma_2) = 0 & \text{for all } \gamma_1,\gamma_2 \in \Gamma. \end{cases}$$

It was proved in [1, Theorem 3.2] that the 2-cocycle $z_1 \smile h$ represents a non-trivial cohomology class in $H^2(\Gamma;\mathbf{C}_\alpha)$ provided that z_1 is not a coboundary and that the $(t-\alpha)$-torsion of the Alexander module is semi-simple i.e. $\tau_\alpha = \mathbf{C}[t^{\pm1}]/(t-\alpha) \oplus \cdots \oplus \mathbf{C}[t^{\pm1}]/(t-\alpha)$. Hence if we suppose that z_1 is not a coboundary then it is clear that a non-abelian representation $\Gamma \to \mathrm{GL}(3,\mathbf{C})$ given by (3) can only exist if the $(t-\alpha)$-torsion τ_α of the Alexander module has a direct summand of the form $\mathbf{C}[t^{\pm1}]/(t-\alpha)^s$, $s \geq 2$.

Representations $\Gamma \to \mathrm{GL}(n,\mathbf{C})$ of this type were studied in [13] where it was shown that the whole structure of the $(t-\alpha)$-torsion of the Alexander module can be recovered. Note that every metabelian representation of Γ factors through the metabelian group $\Gamma'/\Gamma'' \rtimes \mathbf{Z}$.

Let $\alpha \in \mathbf{C}^*$ be a non-zero complex number and $n \in \mathbf{Z}$, $n > 1$. In what follows we consider the cyclic $\mathbf{C}[t^{\pm1}]$-module $\mathbf{C}[t^{\pm1}]/(t-\alpha)^{n-1}$ and the semi-direct product

$$\mathbf{C}[t^{\pm1}]/(t-\alpha)^{n-1} \rtimes \mathbf{Z}$$

where the multiplication is given by $(p_1,n_1)(p_2,n_2) = (p_1 + t^{n_1}p_2, n_1+n_2)$. Let $I_n \in \mathrm{SL}(n)$ and $N_n \in \mathrm{GL}(n)$ denote the identity matrix and the upper triangular Jordan normal form of a nilpotent matrix of degree n respectively. For later use we note the following lemma which follows easily from the Jordan normal form theorem:

3.1 Lemma *Let $\alpha \in \mathbf{C}^*$ be a nonzero complex number and let $\mathbf{C}^n$ be the $\mathbf{C}[t^{\pm 1}]$-module with the action of t^k given by*

$$t^k\,\mathbf{a} = \alpha^k\,\mathbf{a}\,J_n^k \tag{4}$$

where $\mathbf{a} \in \mathbf{C}^n$ and $J_n = I_n + N_n$. Then the $\mathbf{C}[t^{\pm 1}]$-module $\mathbf{C}^n$ is isomorphic to $\mathbf{C}[t^{\pm 1}]/(t-\alpha)^n$.

There is a direct method to construct a reducible metabelian representations of $\mathbf{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbf{Z}$ into $\mathrm{GL}(n,\mathbf{C})$ (see [4, Proposition 3.13]). A direct calculation gives that

$$(\mathbf{a},0) \mapsto \begin{pmatrix} 1 & \mathbf{a} \\ \mathbf{0} & I_{n-1} \end{pmatrix}, \quad (0,1) \mapsto \begin{pmatrix} \alpha & \mathbf{0} \\ \mathbf{0} & J_{n-1}^{-1} \end{pmatrix}$$

defines a faithful representation $\mathbf{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbf{Z} \to \mathrm{GL}(n,\mathbf{C})$.

Therefore, we obtain a reducible, metabelian, non-abelian representation $\tilde{\varrho}\colon \Gamma \to \mathrm{GL}(n,\mathbf{C})$ if the Alexander module $H_1(X_\infty,\mathbf{C})$ has a direct summand of the form $\mathbf{C}[t^{\pm 1}]/(t-\alpha)^s$ with $s \geq n-1 \geq 1$:

$$\begin{aligned} \tilde{\varrho}\colon \Gamma \cong \Gamma' \rtimes \mathbf{Z} \to \Gamma'/\Gamma'' \rtimes \mathbf{Z} \to (\mathbf{C}\otimes \Gamma'/\Gamma'') \rtimes \mathbf{Z} \to \\ \mathbf{C}[t^{\pm 1}]/(t-\alpha)^s \rtimes \mathbf{Z} \to \mathbf{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbf{Z} \to \mathrm{GL}(n,\mathbf{C}) \end{aligned}$$

given by

$$\tilde{\varrho}(\gamma) = \begin{pmatrix} 1 & \tilde{\mathbf{z}}(\gamma) \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} \alpha^{h(\gamma)} & 0 \\ 0 & J_{n-1}^{-h(\gamma)} \end{pmatrix}. \tag{5}$$

It is easy to see that a map $\tilde{\varrho}\colon \Gamma \to \mathrm{GL}(n)$ given by (5) is a homomorphism if and only if $\tilde{\mathbf{z}}\colon \Gamma \to \mathbf{C}^{n-1}$ is a cocycle i.e. for all $\gamma_1,\gamma_2 \in \Gamma$ we have

$$\tilde{\mathbf{z}}(\gamma_1\gamma_2) = \tilde{\mathbf{z}}(\gamma_1) + \alpha^{h(\gamma_1)}\tilde{\mathbf{z}}(\gamma_2) J_{n-1}^{h(\gamma_1)}. \tag{6}$$

For a better description of the cocycle $\tilde{\mathbf{z}}$, we introduce the following notations: for $m,\ k \in \mathbf{Z},\ k \geq 0$, we define

$$h_k(\gamma) := \binom{h(\gamma)}{k} \quad \text{where} \quad \binom{m}{k} := \begin{cases} \frac{m(m-1)\cdots(m-k+1)}{k!} & \text{if } k>0 \\ 1 & \text{if } k=0. \end{cases} \tag{7}$$

It follows directly from the properties of the binomial coefficients that for each $k \in \mathbf{Z},\ k \geq 0$, the cochains $h_k \in C^1(\Gamma;\mathbf{C})$ are defined and verify:

$$\delta h_k + \sum_{i=1}^{k-1} h_i \smile h_{k-i} = 0. \tag{8}$$

3.2 Lemma *Let* $\tilde{\mathbf{z}}\colon \Gamma \to \mathbf{C}^{n-1}$ *be a map verifying* (6) *and let* $\tilde{z}_k\colon \Gamma \to \mathbf{C}_\alpha$, $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_{n-1})$, *denote the components of* $\tilde{\mathbf{z}}$. *Then the cochains* $\tilde{z}_k$, $1 \le k \le n-1$, *satisfy*

$$\delta \tilde{z}_k + \sum_{i=1}^{k-1} h_i \smile \tilde{z}_{k-i} = 0\,.$$

In particular $\tilde{z}_1\colon \Gamma \to \mathbf{C}_\alpha$ *is a cocycle.*

Proof. Note that $h_0 \equiv 1$, $h_1 = h$, $J_{n-1}^m = (I_{n-1} + N_{n-1})^m = \sum_{i\ge 0} \binom{m}{i} N_{n-1}^i$ and $(x_1, \dots, x_{n-1}) J_{n-1}^m = (x_1', x_2', \dots, x_{n-1}')$ where

$$x_k' = \sum_{i=0}^{k-1} \binom{m}{i} x_{k-i} = x_k + \sum_{i=1}^{k-1} \binom{m}{i} x_{k-i}\,.$$

It follows from this formula that $\tilde{\mathbf{z}}(\gamma_1\gamma_2) = \tilde{\mathbf{z}}(\gamma_1) + \alpha^{h(\gamma_1)} \tilde{\mathbf{z}}(\gamma_2) J_{n-1}^{h(\gamma_1)}$ holds if and only if for $k = 1, \dots, n-1$ we have

$$\tilde{z}_k(\gamma_1\gamma_2) = \tilde{z}_k(\gamma_1) + \alpha^{h(\gamma_1)} \tilde{z}_k(\gamma_2) + \sum_{i=1}^{k-1} h_i(\gamma_1)\, \alpha^{h(\gamma_1)} \tilde{z}_{k-i}(\gamma_2)\,.$$

In other words $0 = \delta \tilde{z}_k + \sum_{i=1}^{k-1} h_i \smile \tilde{z}_{k-i}$ holds. □

From now on we will suppose that for $\alpha \in \mathbf{C}^* \setminus \{1\}$ the $(t-\alpha)$-torsion of the Alexander module is cyclic of the form

$$\tau_\alpha = \mathbf{C}[t^{\pm 1}]/(t-\alpha)^{n-1}\,, \qquad \text{where } n \ge 2\,.$$

This is equivalent to the fact that α is a root of the Alexander polynomial $\Delta_K(t)$ of multiplicity $n-1$ and that $\dim H^1(\Gamma; \mathbf{C}_\alpha) = 1$ (see Lemma 2.3). Let us recall also that by Lemma 2.3, the following dimension formulas hold:

$$\dim H^q(\Gamma; \mathbf{C}) = \begin{cases} 1 & \text{for } q = 0, 1; \\ 0 & \text{for } q \ge 2, \end{cases} \tag{9}$$

and

$$\dim H^q(\Gamma; \mathbf{C}_{\alpha^{\pm 1}}) = \begin{cases} 1 & \text{for } q = 1, 2; \\ 0 & \text{for } q \ne 1, 2. \end{cases} \tag{10}$$

3.3 Remark Notice that by Blanchfield-duality the $(t-\alpha^{-1})$-torsion of the Alexander module $H_1(\Gamma;\mathbf{C}[t^{\pm1}])$ is also of the form

$$\tau_{\alpha^{-1}} = \mathbf{C}[t^{\pm1}]/(t-\alpha^{-1})^{n-1}.$$

More precisely, the Alexander polynomial $\Delta_K(t)$ is symmetric and hence α^{-1} is also a root of $\Delta_K(t)$ of multiplicity $n-1$ and $\dim H^1(\Gamma;\mathbf{C}_{\alpha^{-1}}) = 1$.

Let $\tilde\varrho\colon \Gamma\to \mathrm{GL}(n)$ be a representation given by (5) i.e. for all $\gamma\in\Gamma$ we have

$$\tilde\varrho(\gamma) = \begin{pmatrix} 1 & \tilde{\mathbf{z}}(\gamma)\\ 0 & I_{n-1}\end{pmatrix}\begin{pmatrix}\alpha^{h(\gamma)} & 0\\ 0 & J_{n-1}^{-h(\gamma)}\end{pmatrix}.$$

We will say that $\tilde\varrho$ can be *upgraded* to a representation into $\mathrm{GL}(n+1,\mathbf{C})$ if there is a cochain $\tilde z_n\colon \Gamma\to\mathbf{C}_\alpha$ such that the map $\Gamma\to\mathrm{GL}(n+1,\mathbf{C})$ given by

$$\gamma\mapsto \begin{pmatrix} 1 & (\tilde{\mathbf{z}}(\gamma),\tilde z_n(\gamma))\\ 0 & I_n\end{pmatrix}\begin{pmatrix}\alpha^{h(\gamma)} & 0\\ 0 & J_n^{-h(\gamma)}\end{pmatrix}$$

is a representation.

3.4 Lemma *Suppose that the $(t-\alpha)$-torsion of the Alexander module is cyclic of the form $\tau_\alpha = \mathbf{C}[t^{\pm1}]/(t-\alpha)^{n-1}$, $n\geq 2$ and let $\tilde\varrho\colon\Gamma\to\mathrm{GL}(n,\mathbf{C})$ be a representation given by* (5).

Then $\tilde\varrho$ cannot be upgraded to a representation into $\mathrm{GL}(n+1,\mathbf{C})$ *unless $\tilde z_1\colon\Gamma\to\mathbf{C}_\alpha$ is a coboundary.*

Proof. By Lemma 3.1 the $\mathbf{C}[t^{\pm1}]$-module $\mathbf{C}^{n-1}$ with the action given by $t\,\mathbf{a} = \alpha\,\mathbf{a}\,J_{n-1}$ is isomorphic to $\mathbf{C}[t^{\pm1}]/(t-\alpha)^{n-1}$. Hence it follows from the universal coefficient theorem that for $l\geq n-1$ we have:

$$\begin{aligned} H^1(\Gamma;\mathbf{C}[t^{\pm1}]/(t-\alpha)^l) &\cong \mathrm{Hom}_{\mathbf{C}[t^{\pm1}]}\big(H_1(\Gamma;\mathbf{C}[t^{\pm1}]),\mathbf{C}[t^{\pm1}]/(t-\alpha)^l\big)\\ &\cong \mathrm{Hom}_{\mathbf{C}[t^{\pm1}]}\big(\mathbf{C}[t^{\pm1}]/(t-\alpha)^{n-1},\mathbf{C}[t^{\pm1}]/(t-\alpha)^l\big)\\ &\cong (t-\alpha)^{l-n+1}\mathbf{C}[t^{\pm1}]/(t-\alpha)^l \cong \mathbf{C}[t^{\pm1}]/(t-\alpha)^{n-1}. \end{aligned}$$

Hence if $l>n-1$ then every cocycle $\tilde z\colon\Gamma\to\mathbf{C}[t^{\pm1}]/(t-\alpha)^l$, given by $\tilde z(\gamma) = (\tilde z_1(\gamma),\dots,\tilde z_l(\gamma))$ is cohomologous to a cocycle for which the first $l-n+1$ components vanish. This proves the conclusion of the lemma. □

Notice that the unipotent matrices J_n and J_n^{-1} are similar: a direct calculation shows that $P_nJ_nP_n^{-1} = J_n^{-1}$ where $P_n = (p_{ij})$, $p_{ij} = (-1)^j\binom{j}{i}$. The matrix P_n is upper triangular with ±1 in the diagonal and P_n^2 is the identity matrix, and therefore $P_n = P_n^{-1}$.

Hence $\tilde{\varrho}$ is conjugate to a representation $\varrho\colon \Gamma \to \mathrm{GL}(n,\mathbf{C})$ given by

$$\varrho(\gamma) = \begin{pmatrix} \alpha^{h(\gamma)} & z(\gamma) \\ 0 & J_{n-1}^{h(\gamma)} \end{pmatrix} = \begin{pmatrix} \alpha^{h(\gamma)} & z_1(\gamma) & z_2(\gamma) & \dots & z_{n-1}(\gamma) \\ 0 & 1 & h_1(\gamma) & \dots & h_{n-2}(\gamma) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & h_1(\gamma) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \tag{11}$$

where $\mathbf{z} = (z_1, \dots, z_{n-1})\colon \Gamma \to \mathbf{C}^{n-1}$ satisfies

$$\mathbf{z}(\gamma_1\gamma_2) = \alpha^{h(\gamma_1)}\mathbf{z}(\gamma_2) + \mathbf{z}(\gamma_1) J_{n-1}^{h(\gamma_2)}\,.$$

It follows directly that $\mathbf{z}(\gamma) = \tilde{\mathbf{z}}(\gamma) P_{n-1} J_{n-1}^{h(\gamma)}$ and in particular $z_1 = -\tilde{z}_1$.

The same argument as in the proof of Lemma 3.2 shows that the cochains $z_k\colon \Gamma \to \mathbf{C}_\alpha$ verify:

$$\delta z_k + \sum_{i=1}^{k-1} z_i \smile h_{k-i} = 0 \quad \text{for } k = 1, \dots, n-1\,.$$

Therefore, the representation $\varrho\colon \Gamma \to \mathrm{GL}(n,\mathbf{C})$ can be upgraded into a representation $\Gamma \to \mathrm{GL}(n+1,\mathbf{C})$ if and only if $\sum_{i=1}^{n-1} z_i \smile h_{n-i}$ is a coboundary.

Hence we obtain the following:

3.5 Proposition *Suppose that the $(t-\alpha)$-torsion of the Alexander module is cyclic of the form $\tau_\alpha = \mathbf{C}[t^{\pm 1}]/(t-\alpha)^{n-1}$, $n \geq 2$. Let $\tilde{\varrho}, \varrho\colon \Gamma \to \mathrm{GL}(n,\mathbf{C})$ be the representations given by (5) and (11) respectively where $\tilde{z}_1 = -z_1\colon \Gamma \to \mathbf{C}_\alpha$ is a non-principal derivation. Then the representations $\tilde{\varrho}$ and ϱ can not be upgraded to representations $\Gamma \to \mathrm{GL}(n+1,\mathbf{C})$ i.e. the cocycles*

$$\sum_{i=1}^{n-1} h_i \smile \tilde{z}_{n-i} \quad \text{and} \quad \sum_{i=1}^{n-1} z_i \smile h_{n-i}$$

represent nontrivial cohomology classes in $H^2(\Gamma; \mathbf{C}_\alpha)$.

Proof. The proposition follows from Lemma 3.4 and the above considerations. $\square$

4 Cohomological computations

We suppose throughout this section that $K \subset S^3$ is a knot and that the $(t-\alpha)$-torsion of its Alexander module is cyclic of the form $\tau_\alpha = \mathbf{C}[t,t^{-1}]/(t-\alpha)^{n-1}$, $n \geq 2$, where $\alpha \in \mathbf{C}^*$ is a nonzero complex number. Let $\varrho\colon \Gamma \to \mathrm{GL}(n)$ be a representation given by (11) where $z_1\colon\ \Gamma \to \mathbf{C}_\alpha$ is a non-principal derivation:

$$\varrho(\gamma) = \begin{pmatrix} \alpha^{h(\gamma)} & z(\gamma) \\ 0 & J_{n-1}^{h(\gamma)} \end{pmatrix} = \begin{pmatrix} \alpha^{h(\gamma)} & z_1(\gamma) & z_2(\gamma) & \dots & z_{n-1}(\gamma) \\ 0 & 1 & h_1(\gamma) & \dots & h_{n-2}(\gamma) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & h_1(\gamma) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

We choose an n-th root λ of $\alpha = \lambda^n$ and we define a reducible metabelian representation $\varrho_\lambda\colon\ \Gamma \to \mathrm{SL}(n)$ by

$$\varrho_\lambda(\gamma) = \lambda^{-h(\gamma)}\varrho(\gamma) \tag{12}$$

The aim of the following sections is to calculate the cohomological groups of Γ with coefficients in the Lie algebra $\mathfrak{sl}(n)_{\mathrm{Ad}\circ\varrho_\lambda}$. Notice that the action of Γ via $\mathrm{Ad}\circ\varrho$ and $\mathrm{Ad}\circ\varrho_\lambda$ preserve $\mathfrak{sl}(n)$ and coincide since the center of $\mathrm{GL}(n)$ is the kernel of $\mathrm{Ad}\colon\ \mathrm{GL}(n) \to \mathrm{Aut}(\mathfrak{gl}(n))$. Hence we have the following isomorphisms of Γ-modules:

$$\mathfrak{sl}(n)_{\mathrm{Ad}\circ\varrho_\lambda} \cong \mathfrak{sl}(n)_{\mathrm{Ad}\circ\varrho} \quad \text{and} \quad \mathfrak{gl}(n)_{\mathrm{Ad}\circ\varrho} = \mathfrak{sl}(n)_{\mathrm{Ad}\circ\varrho} \oplus \mathbf{C}\, I_n \tag{13}$$

where Γ acts trivially on the center $\mathbf{C}I_n$ of $\mathfrak{gl}(n)$. We will prove the following result:

4.1 Proposition *Let $K \subset S^3$ be a knot and suppose that the $(t-\alpha)$-torsion of the Alexander module of K is of the form $\tau_\alpha = \mathbf{C}[t^{\pm1}]/(t-\alpha)^{n-1}$. Then for the representation $\varrho_\lambda\colon\ \Gamma \to \mathrm{SL}(n)$ we have $H^0(\Gamma;\mathfrak{sl}(n)_{\mathrm{Ad}\circ\varrho_\lambda}) = 0$ and*

$$\dim H^1(\Gamma;\mathfrak{sl}(n)_{\mathrm{Ad}\circ\varrho_\lambda}) = \dim H^2(\Gamma;\mathfrak{sl}(n)_{\mathrm{Ad}\circ\varrho_\lambda}) = n-1\,.$$

Notice that Propositions 4.1 and 2.4 will proof the first part of Theorem 1.1. The proof of Proposition 4.1 will occupy the rest of this section.

Throughout this section we will consider $\mathfrak{gl}(n)$ as a Γ-module via $\mathrm{Ad}\circ\varrho$ and for simplicity we will write $\mathfrak{gl}(n)$ for $\mathfrak{gl}(n)_{\mathrm{Ad}\circ\varrho}$. It follows form Equation (13) that

$$H^*(\Gamma;\mathfrak{gl}(n)) \cong H^*(\Gamma;\mathfrak{sl}(n)) \oplus H^*(\Gamma;\mathbf{C})\,.$$

In order to compute the cohomological groups $H^*(\Gamma, \mathfrak{gl}(n))$ and describe the cocycles, we will construct and use an adequate filtration of the coefficient algebra $\mathfrak{gl}(n)$.

4.1 The setup

Let $(E_1, \ldots, E_n)$ denote the canonical basis of the space of column vectors. Hence $E_i^j := E_i\,{}^tE_j$, $1 \le i, j \le n$, form the canonical basis of $\mathfrak{gl}(n)$.

Note that for $A \in \mathrm{GL}(n)$, $\mathrm{Ad}_A(E_i^j) = (AE_i)({}^tE_jA^{-1})$. The Lie algebra $\mathfrak{gl}(n)$ turns into a Γ-module via $\mathrm{Ad} \circ \varrho$ i.e. for all $\gamma \in \Gamma$ we have

$$\gamma \cdot E_i^j = (\varrho(\gamma)E_i)({}^tE_j\varrho(\gamma^{-1}))\,.$$

Explicitly we have

$$\begin{aligned}\gamma \cdot E_1^1 &= \begin{pmatrix} \alpha^{h(\gamma)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \big(\alpha^{-h(\gamma)}, z_1(\gamma^{-1}), \ldots, z_{n-1}(\gamma^{-1})\big) \\ &= E_1^1 + \alpha^{h(\gamma)} z_1(\gamma^{-1}) E_1^2 + \cdots + \alpha^{h(\gamma)} z_{n-1}(\gamma^{-1}) E_1^n; \end{aligned} \tag{14}$$

for $1 < k \le n$:

$$\gamma \cdot E_1^k = \alpha^{h(\gamma)} E_1^k + \alpha^{h(\gamma)} h_1(\gamma^{-1}) E_1^{k+1} + \cdots + \alpha^{h(\gamma)} h_{n-k}(\gamma^{-1}) E_1^n; \tag{15}$$

$$\gamma \cdot E_k^1 = \begin{pmatrix} z_{k-1}(\gamma) \\ h_{k-2}(\gamma) \\ \vdots \\ h_1(\gamma) \\ 1 \\ 0 \\ \vdots \end{pmatrix} \big(\alpha^{-h(\gamma)}, z_1(\gamma^{-1}), \ldots, z_{n-1}(\gamma^{-1})\big) \tag{16}$$

and for $1 < i, j \le n$:

$$\gamma \cdot E_i^j = \begin{pmatrix} z_{i-1}(\gamma) \\ h_{i-2}(\gamma) \\ \vdots \\ h_1(\gamma) \\ 1 \\ 0 \\ \vdots \end{pmatrix} \big(0, \ldots, 0, 1, h_1(\gamma^{-1}), \ldots, h_{n-j}(\gamma^{-1})\big)\,. \tag{17}$$

For a given family $(X_i)_{i\in I}$, $X_i \in \mathfrak{gl}(n)$, we let $\langle X_i | i \in I\rangle \subset \mathfrak{gl}(n)$ denote the subspace of $\mathfrak{gl}(n)$ generated by the family.

4.2 Remark A first consequence of these calculations is that if $c \in C^1(\Gamma;\mathbf{C})$ is a cochain, then for $2 \le i \le n$ and $1 \le j \le n$ we have:

$$\delta^{\mathfrak{gl}}(cE_i^j) = (\delta c)E_i^j + (h_1 \smile c)E_{i-1}^j + \cdots + (h_{i-2} \smile c)E_2^j + (z_{i-1} \smile c)E_1^j + x$$

where $x\colon \Gamma\times\Gamma \to \langle E_k^l \mid 1 \le k \le i,\ j < l \le n\rangle$ is a 2-cochain. Here $\delta^{\mathfrak{gl}}$ and δ denote the coboundary operators of $C^1(\Gamma;\mathfrak{gl}(n))$ and $C^1(\Gamma;\mathbf{C})$ respectively.

In what follows we will also make use of the following Γ-modules: for $0 \le i \le n-1$, we define $C(i) = \langle E_k^l \mid 1 \le k \le n,\, n-i \le l \le n\rangle$. We have

$$C(i) = \left\{ \begin{pmatrix} 0 & \cdots & 0 & c_{1,n-i} & \cdots & c_{1,n} \\ 0 & \cdots & 0 & c_{2,n-i} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & c_{n-1,n-i} & \cdots & c_{n-1,n} \\ 0 & \cdots & 0 & c_{n,n-i} & \cdots & c_{n,n} \end{pmatrix} : c_{i,j} \in \mathbf{C} \right\} \tag{18}$$

and $\mathfrak{gl}(n) = C(n-1) \supset C(n-2) \supset \cdots \supset C(0) = \langle E_1^n, \ldots, E_n^n\rangle \supset C(-1) = 0$.

We will denote by $X + C(i) \in C(k)/C(i)$ the class represented by $X \in C(k)$, $0 \le i < k \le n-1$.

4.2 Cohomology with coefficients in $C(i)$

The aim of this subsection is to prove that for $0 \le i \le n-2$ the cohomology groups $H^*(\Gamma;C(i))$ vanish (see Proposition (4.7)). First we will prove this for $i=0$ and in order to conclude we will apply the isomorphism $C(0) \cong C(i)/C(i-1)$ (see Lemma 4.5). Finally Lemma 4.6 permits us to compute a certain Bockstein operator.

4.3 Lemma *The vector space $\langle E_1^n\rangle$ is a submodule of $C(0)$ and thus of $\mathfrak{gl}(n) = C(n-1)$ and we have*

$$H^0(\Gamma;\langle E_1^n\rangle) = 0,\ \dim H^1(\Gamma;\langle E_1^n\rangle) = \dim H^2(\Gamma;\langle E_1^n\rangle) = 1.$$

More precisely, the cocycles $z_1\, E_1^n \in Z^1(\Gamma;\langle E_1^n\rangle)$ and

$$\Big(\sum_{i=1}^{n-1} z_i \smile h_{n-i}\Big)\, E_1^n \in Z^2(\Gamma;\langle E_1^n\rangle)$$

represent generators of $H^1(\Gamma;\langle E_1^n\rangle)$ and $H^2(\Gamma;\langle E_1^n\rangle)$ respectively.

Proof. The isomorphism $\langle E_1^n \rangle \cong \mathbf{C}_\alpha$ and Lemma 2.3 imply the dimension formulas. The form of the generating cocycles follows from the isomorphism $\langle E_1^n \rangle \cong \mathbf{C}_\alpha$ and Proposition 3.5. $\square$

4.4 Lemma *The Γ-module $C(0)/\langle E_1^n \rangle$ is isomorphic to $\mathbf{C}[t^{\pm 1}]/(t-1)^{n-1}$. In particular, we obtain:*

1. *for $q = 0, 1$ $\dim H^q\big(\Gamma; C(0)/\langle E_1^n \rangle\big) = 1$ and $H^2\big(\Gamma; C(0)/\langle E_1^n \rangle\big) = 0$,*
2. *the vector E_2^n represents a generator of $H^0\left(\Gamma; C(0)/\langle E_1^n \rangle\right)$ and the cochain $\bar{v}_1 \colon \Gamma \to C(0)$ given by*

$$\bar{v}_1(\gamma) = h_1(\gamma) E_n^n + h_2(\gamma) E_{n-1}^n + \cdots + h_{n-2}(\gamma) E_2^n$$

represents a generator of $H^1(\Gamma; C(0)/\langle E_1^n \rangle)$.

Proof. First notice that $C(0)/\langle E_1^n \rangle$ is a $(n-1)$-dimensional vector space. More precisely, a basis of this space is represented by the elements

$$E_n^n, E_{n-1}^n, \ldots, E_2^n \,.$$

It follows from (17) that the action of Γ on $C(0)/\langle E_1^n \rangle$ factors through $h\colon \Gamma \to \mathbf{Z}$. More precisely, we have for all $\gamma \in \Gamma$ such that $h(\gamma) = 1$ and for all $0 \le l \le n-1$

$$\gamma \cdot E_{n-l}^n = E_{n-l}^n + E_{n-l-1}^n$$

Here we used the fact that if $h(\gamma) = 1$ then $h_i(\gamma) = 0$ for all $2 \le i \le n-1$.

On the other hand

$$\big(1 = (t-1)^0, (t-1), \ldots, (t-1)^{n-2}\big)$$

represents a basis of $\mathbf{C}[t^{\pm 1}]/(t-1)^{n-1}$ and we have for all $\gamma \in \Gamma$ such that $h(\gamma) = 1$:

$$\gamma \cdot (t-1)^l = (t-1)^l + (t-1)^{l+1} + p$$

where $p \in (t-1)^{n-1}\mathbf{C}[t^{\pm 1}]$ and $0 \le l \le n-2$. Hence the bijection

$$\varphi\colon\ \{(t-1)^l \mid 0 \le l \le n-2\} \to \{E_{n-l}^n \mid 0 \le l \le n-2\}$$

given by $\varphi\colon\ (t-1)^l \mapsto E_{n-l}^n$, $0 \le l \le n-2$, induces an isomorphism of Γ-modules

$$\varphi\colon\ \mathbf{C}[t^{\pm 1}]/(t-1)^{n-1} \xrightarrow{\cong} C(0)/\langle E_1^n \rangle \,.$$

Now, the first assertion follows from Lemma 2.3.

Moreover, it follows from the above considerations that E_2^n represents a generator of $H^0(\Gamma; C(0)/\langle E_1^n\rangle)$. To prove the second assertion consider the following short exact sequence

$$0 \to \mathbf{C}[t^{\pm 1}]/(t-1)^{n-2} \xrightarrow{(t-1)\cdot} \mathbf{C}[t^{\pm 1}]/(t-1)^{n-1} \to \mathbf{C} \to 0$$

which gives the following long exact sequence in cohomology:

$$\begin{gathered} 0 \to H^0(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-2}) \xrightarrow{\cong} H^0(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-1}) \to \\ H^0(\Gamma; \mathbf{C}) \xrightarrow{\beta^0} H^1(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-2}) \to \\ H^1(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-1}) \xrightarrow{\cong} H^1(\Gamma; \mathbf{C}) \to H^2(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-2}) = 0\,. \end{gathered}$$

The isomorphisms and the vanishing of $H^2(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-2})$ follow directly from Lemma 2.3.

Hence the Bockstein operator β^0 is an isomorphism: the element $e_0 = 1 \in \mathbf{C}[t^{\pm 1}]/(t-1)^{n-1}$ projects onto a generator of $H^0(\Gamma; \mathbf{C})$ and if δ^{n-1} denotes the coboundary operator of $C^*(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-1})$ we obtain:

$$\begin{aligned} \delta^{n-1}(e_0)(\gamma) &= (\gamma - 1)\cdot e_0 \\ &= h_1(\gamma)e_1 + h_2(\gamma)e_2 + \cdots + h_{n-2}(\gamma)e_{n-1} \\ &= (t-1)\cdot \big(h_1(\gamma)e_0 + h_2(\gamma)e_1 + \cdots + h_{n-2}(\gamma)e_{n-2}\big)\,. \end{aligned}$$

Hence the cocycle $\gamma \mapsto h_1(\gamma)e_0 + h_2(\gamma)e_1 + \cdots + h_{n-2}(\gamma)e_{n-2}$ represents a generator of $H^1(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-1)^{n-2})$. To conclude, recall that the isomorphism $\mathbf{C}[t^{\pm 1}]/(t-1)^{n-1} \cong C(0)/\langle E_1^n\rangle$ is induced by the map $\varphi\colon\ e_l \mapsto E_{n-l}^n$, $0 \le l \le n-2$. □

4.5 Lemma *For $i \in \mathbf{Z}$, $0 \le i \le n-3$, the Γ-module $C(i+1)/C(i)$ is isomorphic to $C(0)$.*

Proof. It follows from (17) that, for all $i \in \mathbf{Z}$, $0 \le i \le n-2$, the bijection

$$\phi\colon\ \{E_{n-j}^{n-(i+1)} + C(i) \mid 0 \le j \le n-1\} \to \{E_{n-j}^n \mid 0 \le j \le n-1\}$$

given by $\phi(E_{n-j}^{n-(i+1)} + C(i)) = E_{n-j}^n$ induces an isomrphism of Γ-modules $\phi : C(i+1)/C(i) \to C(0)$. □

Let us recall the definition of the cochains $h_i \in C^1(\Gamma; \mathbf{C})$, given by $h_i(\gamma) = \binom{h(\gamma)}{i}$ (see Equation (7)). Recall also that for $1 \le i \le n-1$ the cochains $h_i \in C^1(\Gamma; \mathbf{C})$ verify Equation (8):

$$\delta h_i + \sum_{j=1}^{i-1} h_j \smile h_{i-j} = 0.$$

4.6 Lemma *Let $\delta^{\mathfrak{gl}}$ denote the coboundary operator of $C^*(\Gamma;\mathfrak{gl}(n))$. Then for all $0 \le k \le n-2$ there exists a cochain $x_{k-1} \in C^2(\Gamma; C(k-1))$ such that*

$$\delta^{\mathfrak{gl}}\big(\sum_{i=2}^{n} h_{n-i+1}E_i^{n-k}\big) = \big(\sum_{i=1}^{n-1} z_i \smile h_{n-i}\big)E_1^{n-k} + x_{k-1}$$

Proof. Equation (17) and Remark 4.2 imply that

$$\begin{aligned}&\delta^{\mathfrak{gl}}(h_{n-i+1}E_i^{n-k}) = \\ &\qquad z_{i-1} \smile h_{n-i+1}\, E_1^{n-k} + \sum_{l=2}^{i-1} h_{i-l} \smile h_{n-i+1}\, E_l^{n-k} + \delta h_{n-i+1}\, E_i^{n-k} + x_{i,k-1}\end{aligned}$$

where $x_{i,k-1} \in C^2(\Gamma; C(k-1))$ and δ is the boundary operator of $C^*(\Gamma;\mathbf{C})$. Therefore,

$$\begin{aligned}\delta^{\mathfrak{gl}}(\sum_{i=2}^{n} h_{n-i+1}E_i^{n-k}) = \big(\sum_{i=2}^{n} z_{i-1} \smile h_{n-i+1}\big)E_1^{n-k} + \sum_{i=2}^{n}\sum_{l=2}^{i-1} h_{i-l} \smile h_{n-i+1}\, E_l^{n-k} \\ + \sum_{i=2}^{n} \delta h_{n-i+1}\, E_i^{n-k} + x_{k-1}\,.\end{aligned}$$

where $x_{k-1} = \sum_{i=2}^{n} x_{i,k-1} \in C^2(\Gamma; C(k-1))$. A direct calculation gives that

$$\begin{aligned}\sum_{i=2}^{n}\sum_{l=2}^{i-1} h_{i-l} \smile h_{n-i+1}\, E_l^{n-k} &= \sum_{l=2}^{n-1}\sum_{i=l+1}^{n} h_{i-l} \smile h_{n-i+1}\, E_l^{n-k} \\ &= \sum_{l=2}^{n-1}\Big(\sum_{i=1}^{n-l} h_i \smile h_{n-l+1-i}\Big)\, E_l^{n-k}\,.\end{aligned}$$

Thus

$$\begin{aligned}\delta^{\mathfrak{gl}}(h_{n-i+1}E_i^{n-k}) = \big(\sum_{i=1}^{n-1} z_i \smile h_{n-i}\big)E_1^{n-k} \\ + \delta h_1\, E_n^{n-k} + \sum_{i=1}^{n-2}\big(\delta h_{n-i} + \sum_{l=1}^{n-i-1} h_l \smile h_{n-i-l}\big)\, E_i^{n-k} + x_{k-1}\,.\end{aligned}$$

Now $\delta h_1 = 0$ and by (8) we have $\delta h_{n-i} + \sum_{l=1}^{n-i} h_l \smile h_{n-i+1-l} = 0$. Hence we obtain the claimed formula. $\square$

4.7 Proposition *For all $i \in \mathbf{Z}$, $0 \le i \le n-2$ and $q \ge 0$ we have*

$$H^q(\Gamma; C(i)) = 0.$$

Proof. For $q \geq 3$ we have $H^q(\Gamma; C(i)) = 0$ since the knot exterior X has the homotopy type of a 2-dimensional complex. We start by proving the result for $i = 0$. Consider the short exact sequence

$$0 \to \langle E_1^n \rangle \rightarrowtail C(0) \twoheadrightarrow C(0)/\langle E_1^n \rangle \to 0. \tag{19}$$

As the $\mathbf{C}[t^{\pm 1}]$-modules $\langle E_1^n \rangle$ and $\mathbf{C}_\alpha \cong \mathbf{C}[t^{\pm 1}]/(t - \alpha)$ are isomorphic, the sequence (19) gives us a long exact sequence in cohomology:

$$\begin{aligned} 0 = H^0(\Gamma; \langle E_1^n \rangle) \to H^0(\Gamma; C(0)) \to H^0(\Gamma; C(0)/\langle E_1^n \rangle) \xrightarrow{\beta_0^0} \\ H^1(\Gamma; \langle E_1^n \rangle) \to H^1(\Gamma; C(0)) \to H^1(\Gamma; C(0)/\langle E_1^n \rangle) \xrightarrow{\beta_0^1} \\ H^2(\Gamma; \langle E_1^n \rangle) \to H^2(\Gamma; C(0)) \to H^2(\Gamma; C(0)/\langle E_1^n \rangle) \to 0\,. \end{aligned}$$

Here, for $q = 0, 1$, we denoted by $\beta_0^q : H^q(\Gamma; C(0)/\langle E_1^n \rangle) \to H^{q+1}(\Gamma; \langle E_1^n \rangle)$ the Bockstein homomorphism. By Lemma 4.4, E_2^n represents a generator of $H^0(\Gamma; C(0)/\langle E_1^n \rangle)$, so

$$\begin{aligned} \beta_0^0(E_2^n)(\gamma) &= (\gamma - 1) \cdot (E_2^n) \\ &= \gamma \cdot E_2^n - E_2^n = z_1(\gamma) E_1^n. \end{aligned}$$

By Lemma 4.3 $z_1 E_1^n$ is a generator of $H^1(\Gamma; \langle E_1^n \rangle)$, and by Lemma 4.4 $\dim H^0(\Gamma; C(0)/\langle E_1^n \rangle) = 1 = \dim H^1(\Gamma; \langle E_1^n \rangle)$, thus β_0^0 is an isomorphism. Consequently $H^0(\Gamma; C(0)) = 0$ as $H^0(\Gamma; \langle E_1^n \rangle) = 0$ by Lemma 4.3.

Now by Lemma 4.4, the cochain $\bar{v}_1 \colon \Gamma \to C(0)$ given by

$$\bar{v}_1(\gamma) = h_1(\gamma) E_n^n + h_2(\gamma) E_{n-1}^n + \cdots + h_{n-1}(\gamma) E_2^n$$

represents a generator of $H^1(\Gamma; C(0)/\langle E_1^n \rangle)$ and by Lemma 4.6

$$\beta_0^1 \big(h_1 E_n^n + h_2 E_{n-1}^n + \cdots + h_{n-1} E_2^n \big) = \Big(\sum_{i=1}^{n-1} z_i \smile h_{n-i} \Big) E_1^n\,.$$

Moreover, by Proposition 3.5 the cocycle $\big(\sum_{i=1}^{n-1} z_i \smile h_{n-i} \big) E_1^n$ represents a generator of $H^2(\Gamma; \langle E_1^n \rangle)$. Thus β_0^1 is an isomorphism and $H^q(\Gamma; C(0)) = 0$ for $q = 1, 2$.

Now suppose that $H^q(\Gamma; C(i_0)) = 0$ for $0 \leq i_0 \leq n - 3$, $q = 0, 1, 2$ and consider the following short exact sequence of Γ-modules:

$$0 \to C(i_0) \rightarrowtail C(i_0 + 1) \twoheadrightarrow C(i_0 + 1)/C(i_0) \to 0\,. \tag{20}$$

This sequence induces a long exact sequence in cohomology

$$
\begin{aligned}
0 \to H^0(\Gamma; C(i_0)) \to H^0(\Gamma; C(i_0+1)) \to H^0\left(\Gamma; C(i_0+1)/C(i_0)\right) \to \\
H^1(\Gamma; C(i_0)) \to H^1(\Gamma; C(i_0+1)) \to H^1\left(\Gamma; C(i_0+1)/C(i_0)\right) \to \\
H^2(\Gamma; C(i_0)) \to H^2(\Gamma; C(i_0+1)) \to H^2\left(\Gamma; C(i_0+1)/C(i_0)\right) \to 0\,.
\end{aligned}
$$

Using the hypothesis, we conclude that the groups $H^q(\Gamma; C(i_0+1))$ and $H^q\left(\Gamma; C(i_0+1)/C(i_0)\right)$ are isomorphic for $q = 0, 1, 2$. By Lemma 4.5, we obtain $H^q(\Gamma; C(i_0+1)) \cong H^q(\Gamma; C(0)) = 0$ for $q = 0, 1, 2$. □

4.3 Cohomology with coefficients in $\mathfrak{gl}(n)$

In this subsection we will prove Proposition 4.1.

Proof of Proposition 4.1. In order to compute the dimensions of the cohomology groups $H^*(\Gamma; \mathfrak{gl}(n))$, we consider the short exact sequence

$$
0 \to C(n-2) \rightarrowtail C(n-1) = \mathfrak{gl}(n) \twoheadrightarrow \mathfrak{gl}(n)/C(n-2) \to 0\,. \tag{21}
$$

The sequence (21) gives rise to the following long exact cohomology sequence:

$$
\begin{aligned}
0 \to H^0(\Gamma; \mathfrak{gl}(n)) \to H^0(\Gamma; \mathfrak{gl}(n)/C(n-2)) \to H^1(\Gamma; C(n-2)) \to \\
H^1(\Gamma; \mathfrak{gl}(n)) \to H^1(\Gamma; \mathfrak{gl}(n)/C(n-2)) \to H^2\left(\Gamma; C(n-2)\right) \to \\
H^2(\Gamma; \mathfrak{gl}(n)) \to H^2(\Gamma; \mathfrak{gl}(n)/C(n-2)) \to 0\,.
\end{aligned}
$$

As $H^q(\Gamma; C(n-2)) = 0$ we conclude that

$$
H^q(\Gamma; \mathfrak{gl}(n)) \cong H^q(\Gamma; \mathfrak{gl}(n)/C(n-2))\,.
$$

It remains to understand the quotient $\mathfrak{gl}(n)/C(n-2)$.

Clearly the vectors $E^1_n, \dots, E^1_1$ represent a basis of $\mathfrak{gl}(n)/C(n-2)$ and there exists a Γ-module M such that the following sequence

$$
0 \to \left\langle E^1_1 + C(n-2) \right\rangle \rightarrowtail \mathfrak{gl}(n)/C(n-2) \twoheadrightarrow M \to 0 \tag{22}
$$

is exact. Now the sequence (22) induces the following exact cohomology sequence:

$$
\begin{aligned}
0 \to H^0(\Gamma; \left\langle E^1_1 + C(n-2) \right\rangle) \to H^0(\Gamma; \mathfrak{gl}(n)/C(n-2)) \to H^0(\Gamma; M) \to \\
H^1(\Gamma; \left\langle E^1_1 + C(n-2) \right\rangle) \to H^1(\Gamma; \mathfrak{gl}(n)/C(n-2)) \to H^1(\Gamma; M) \to \\
H^2\left(\Gamma; \left\langle E^1_1 + C(n-2) \right\rangle\right) \to H^2(\Gamma; \mathfrak{gl}(n)/C(n-2)) \to H^2(\Gamma; M) \to 0\,.
\end{aligned} \tag{23}
$$

Observe that the action of Γ on $\langle E_1^1 + C(n-2)\rangle$ is trivial. Therefore, $\langle E_1^1 + C(n-2)\rangle$ and $\mathbf{C}$ are isomorphic Γ-modules. By Lemma 2.3 we obtain

$$\dim H^q(\Gamma; \big\langle E_1^1 + C(n-2)\big\rangle) = 1 \quad \text{for } q = 0, 1$$

and $H^2(\Gamma; \langle E_1^1 + C(n-2)\rangle) = 0$.

To complete the proof we will make use of Lemma 4.8, which states that the Γ-module M is isomorphic to $\mathbf{C}[t^{\pm 1}]/(t-\alpha^{-1})^{n-1}$. Recall that Lemma 2.3 implies that $H^0(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-\alpha^{-1})^{n-1}) = 0$ and

$$\dim H^q(\Gamma; \mathbf{C}[t^{\pm 1}]/(t-\alpha^{-1})^{n-1}) = n-1, \quad \text{for } q = 1, 2.$$

Therefore, sequence (23) gives:

$$H^q(\Gamma; \mathfrak{gl}(n)) \cong H^q(\Gamma; \mathfrak{gl}(n)/C(n-2)) \cong \begin{cases} H^0(\Gamma; \mathbf{C}) & \text{for } q = 0; \\ H^2(\Gamma; M) & \text{for } q = 2 \end{cases}$$

and the short exact sequence:

$$0 \to H^1(\Gamma; \mathbf{C}) \rightarrowtail H^1(\Gamma; \mathfrak{gl}(n)/C(n-2)) \cong H^1(\Gamma; \mathfrak{gl}(n)) \twoheadrightarrow H^1(\Gamma; M) \to 0\,.$$

□

4.8 Lemma *The Γ-module M is isomorphic to $\mathbf{C}[t^{\pm 1}]/(t-\alpha^{-1})^{n-1}$. Consequently*

$$H^0(\Gamma; M) = 0, \quad \dim H^q(\Gamma; M) = n-1, \ q = 0, 1.$$

Proof of Lemma 4.8. The proof is similar to the proof of Lemma 4.4. As a $\mathbf{C}$-vector space the dimension of M is $n-1$ and a basis is given by $\left(\overline{E_n^1}, \dots, \overline{E_2^1}\right)$ where $\overline{E}_i^1 = E_i^1 + C(n-2) \in M$ is the class represented by E_i^1, $2 \le i \le n$. In order to prove that M is isomorphic to $\mathbf{C}[t^{\pm 1}]/(t-\alpha^{-1})^{n-1}$ observe that by (16)

$$\gamma \cdot E_k^1 = \alpha^{-h(\gamma)}\big(E_k^1 + h_1(\gamma)E_{k-1}^1 + \cdots + h_{k-2}(\gamma)E_2^1\big) + X_k$$

where $X_k \in E_1^1 + C(n-2)$. Therefore, the action of Γ on M factors through $h\colon \Gamma \to \mathbf{Z}$. More precisely, we have for all $\gamma \in \Gamma$ such that $h(\gamma) = 1$

$$\gamma \cdot \overline{E}_k^1 = \alpha^{-1}(\overline{E}_k^1 + \overline{E}_{k-1}^1)\,.$$

On the other hand $e_l = \big(\alpha(t-\alpha^{-1})\big)^l$, $0 \le l \le n-2$, represents a basis of $\mathbf{C}[t^{\pm 1}]/(t-\alpha^{-1})^{n-1}$ and we have for all $\gamma \in \Gamma$ such that $h(\gamma) = 1$:

$$\gamma \cdot e_l = \alpha^{-1}(e_l + e_{l+1}) + p \text{ where } p \in (t-\alpha^{-1})^{n-1}\mathbf{C}[t^{\pm 1}].$$

Hence the bijection $\psi\colon \{e_l \mid 0 \le l \le n-2\} \to \{\overline{E}_k^1 \mid 2 \le k \le n\}$ given by $\varphi\colon e_l \mapsto \overline{E}_{n-l}^1$, $0 \le l \le n-2$, induces an isomorphism of Γ-modules $\psi\colon \mathbf{C}[t^{\pm 1}]/(t-\alpha^{-1})^{n-1} \xrightarrow{\cong} M$.

Finally, the dimension equations follow from Lemma 2.3 and Remark 3.3. □

We obtain immediately that under the hypotheses of Proposition 4.1 the representation ϱ_λ is a smooth point of the representation variety $R_n(\Gamma)$. This proves the first part of Theorem 1.1.

4.9 Proposition *Let K be a knot in the 3-sphere S^3. If the $(t-\alpha)$-torsion τ_α of the Alexander module is cyclic of the form $\mathbf{C}[t,t^{-1}]/(t-\alpha)^{n-1}$, $n \ge 2$, then the representation ϱ_λ is a smooth point of the representation variety $R_n(\Gamma)$; it is contained in a unique (n^2+2n-2)-dimensional component R_{ϱ_λ} of $R_n(\Gamma)$.*

Proof. By Proposition 2.4 and Proposition 4.1, the representation ϱ_λ is contained in a unique component R_{ϱ_λ} of dimension (n^2+n-2). Moreover,

$$\begin{aligned}\dim Z^1(\Gamma;\mathfrak{sl}(n)) &= \dim H^1(\Gamma;\mathfrak{sl}(n)) + \dim B^1(\Gamma;\mathfrak{sl}(n))\\ &= (n-1)+(n^2-1)\\ &= n^2+n-2\,.\end{aligned}$$

Hence the representation ϱ_λ is a smooth point of $R_n(\Gamma)$ which is contained in an unique (n^2+n-2)-dimensional component R_{ϱ_λ}. □

For a later use, we describe more precisely the derivations $v_k\colon \Gamma \to \mathfrak{sl}(n)$, $1 \le k \le n-1$, which represent a basis of $H^1(\Gamma;\mathfrak{sl}(n))$.

4.10 Corollary *There exists cochains $z_1^-,\cdots,z_{n-1}^- \in C^1(\Gamma;\mathbf{C}_{\alpha^{-1}})$ such that $\delta z_k^- + \sum_{i=1}^{k-1} h_i \smile z_{k-i}^- = 0$ for $k = 1,\dots,n-1$ and $z_1^-\colon \Gamma \to \mathbf{C}_\alpha^{-1}$ is a non-principal derivation.*

Moreover, there exist cochains $g_k\colon \Gamma \to \mathbf{C}$ and $x_k\colon \Gamma \to \mathbf{C}(n-2)$, $1 \le k \le n-1$, such that the cochains $v_k\colon \Gamma \to \mathfrak{sl}(n)$ given by

$$v_k = g_k E_1^1 + z_k^- E_2^1 + \cdots + z_1^- E_{k+1}^1 + x_k$$

are cocycles and represent a basis of $H^1(\Gamma;\mathfrak{sl}(n))$.

Proof. Recall that the vector space M admits as a basis the family $\left(\overline{E}_n^1,\dots,\overline{E}_2^1\right)$ and that it is isomorphic to $\mathbf{C}[t^{\pm1}]/(t-\alpha^{-1})^{n-1}$. Moreover it

is easily seen that M is isomorphic to the Γ-module of column vectors $\mathbf{C}^{n-1}$ where the action is given by $t^k a = \alpha^{-k} J_{n-1}^k a$. Hence a cochain $\mathbf{z}^-\colon\ \Gamma \to M$ with coordinates $\mathbf{z}^- = {}^t(z_{n-1}^-, \cdots, z_1^-)$ is a cocycle in $Z^1(\Gamma; M)$ if and only if for all $\gamma_1, \gamma_2 \in \Gamma$

$$\mathbf{z}^-(\gamma_1\gamma_2) = \mathbf{z}^-(\gamma_1) + \alpha^{-h(\gamma_1)} J_{n-1}^{h(\gamma_1)} \mathbf{z}^-(\gamma_2).$$

It follows, as in the proof of Lemma 3.2, that this is equivalent to

$$z_k^-(\gamma_1\gamma_2) = z_k^-(\gamma_1) + \alpha^{-h(\gamma_1)} z_k^-(\gamma_2) + \sum_{i=1}^{k-1} h_i(\gamma_1)\alpha^{-h(\gamma_1)} z_{k-i}^-(\gamma_2).$$

In other words, for $1 \le k \le n-1$,

$$0 = \delta z_k^- + \sum_{i=1}^{k-1} h_i \smile z_{k-i}^-.$$

By Remark 3.3, if $z_1^- \in Z^1(\Gamma; \mathbf{C}_{\alpha^{-1}})$ is a non-principal derivation, there exist cochains $z_k^-\colon\ \Gamma \to \mathbf{C}_{\alpha^{-1}}$, $2 \le k \le n-1$, such that

$$0 = \delta z_k^- + \sum_{i=1}^{k-1} h_i \smile z_{k-i}^- .$$

Consequently, as $\dim H^1(\Gamma; M) = n-1$, the cochains

$$\mathbf{z}_k^- = z_k^- \overline{E}_2^1 + \cdots + z_1^- \overline{E}_{k+1}^1, \quad 1 \le k \le n-1,$$

represent a basis of $H^1(\Gamma; M)$. The proof is completed by noticing that the projection $H^1(\Gamma; \mathfrak{gl}(n)) \to H^1(\Gamma; M)$ restricts to an isomorphism between $H^1(\Gamma; \mathfrak{sl}(n))$ and $H^1(\Gamma; M)$. $\square$

5 Irreducible $\mathrm{SL}(n)$ representations

This section will be devoted to the proof of the last part of Theorem 1.1. At first, we proved that the representation ϱ_λ is a smooth point of $R_n(\Gamma)$ which is contained in a unique $(n^2+n-2)-$dimensional component R_{ϱ_λ}. Then, to prove the existence of irreducible representations in that component, we will make use of Corollary 4.10 and Burnside's theorem on matrix algebras.

Proof of the last part of Theorem 1.1. To prove that the component R_{ϱ_λ} contains irreducible non metabelian representations, we will generalize the argument given in [3] for $n=3$.

Let $\Gamma = \langle S_1,\dots,S_n |\ W_1,\dots,W_{n-1}\rangle$ be a Wirtinger presentation of the knot group. Modulo conjugation of the representation ϱ_λ, we can assume that $z_1(S_1)=\ldots=z_{n-1}(S_1)=0$. This conjugation corresponds to adding a coboundary to the cochains z_i, $1\le i\le n-1$. We will also assume that the second Wirtinger generator S_2 verifies $z_1(S_2)=b_1\neq 0 = z_1(S_1)$. This is always possible since z_1 is not a coboundary. Hence

$$\varrho_\lambda(S_1) = \alpha^{-1/n}\left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & J_{n-1}\end{array}\right) \quad \text{and} \quad \varrho_\lambda(S_2) = \alpha^{-1/n}\left(\begin{array}{c|c} \alpha & b \\ \hline 0 & J_{n-1}\end{array}\right)$$

where $b=(b_1,\dots,b_{n-1})$ with $b_1\in\mathbf{C}^*$ and $b_i = z_i(S_2)\in\mathbf{C}$ for $2\le i\le n-1$.

Let $v_{n-1}\in Z^1(\Gamma;\mathfrak{sl}(n))$ be a cocycle such that:

$$v_{n-1} = g_{n-1}E_1^1 + z_1^- E_n^1 + z_2^- E_{n-1}^1 + \ldots + z_{n-1}^- E_2^1 + x_{n-1}$$

given by Corollary 4.10. Up to adding a coboundary to the cocycle z_1^- we assume that $z_1^-(S_1)=0$. Notice that, by Lemma 5.5 of [3], $z_1^-(S_2)\neq 0$.

Let ρ_t be a deformation of ϱ_λ with leading term v_{n-1}:

$$\rho_t = \big(I_n + t\, v_{n-1} + o(t)\big)\varrho_\lambda\,, \text{ where } \lim_{t\to 0}\frac{o(t)}{t} = 0\,.$$

We may apply the following lemma (whose proof is completely analogous to that of Lemma 5.3 in [3]) to this deformation for $A(t)=\rho_t(S_1)$.

5.1 Lemma *Let $\rho_t\colon\Gamma\to\mathrm{SL}(n)$ be a curve in $R_n(\Gamma)$ with $\rho_0=\varrho_\lambda$. Then there exists a curve C_t in $\mathrm{SL}(n)$ such that $C_0=I_n$ and*

$$\mathrm{Ad}_{C_t}\circ\rho_t(S_1) = \begin{pmatrix} a_{11}(t) & 0 & \dots & 0 \\ 0 & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}(t) & \dots & a_{nn}(t)\end{pmatrix}$$

for all sufficiently small t.

Therefore, we may suppose that $a_{n1}(t)=0$, and since

$$a_{n1}(t) = t\lambda^{n-1}\left(z_1^-(S_1)+\delta c(S_1)\right)+o(t)\,, \quad \text{for } c\in\mathbf{C},$$

it follows that

$$a_{n1}'(0) = \lambda^{n-1}(z_1^-(S_1) + (\alpha^{-1}-1)c) = 0$$

and hence $c = 0$. For $B(t) = \rho_t(S_2)$, we obtain $b'_{n1}(0) = \lambda^{n-1} z_1^-(S_2) \neq 0$. Hence, we can apply the following technical lemma (whose proof will be postponed to the end of this section).

5.2 Lemma *Let $A(t) = (a_{ij}(t))_{1\leq i,j\leq n}$ and $B(t) = (b_{ij}(t))_{1\leq i,j\leq n}$ be matrices depending analytically on t such that*

$$A(t) = \left(\begin{array}{c|c} a_{11}(t) & 0 \\ \hline 0 & A_{11}(t) \end{array}\right), \quad A(0) = \varrho_\lambda(S_1) = \alpha^{-1/n} \left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & J_{n-1} \end{array}\right)$$

and

$$B(0) = \varrho_\lambda(S_2) = \alpha^{-1/n} \left(\begin{array}{c|c} \alpha & b \\ \hline 0 & J_{n-1} \end{array}\right).$$

If the first derivative $b'_{n1}(0) \neq 0$ then for sufficiently small t, $t \neq 0$, the matrices $A(t)$ and $B(t)$ generate the full matrix algebra $M(n, \mathbf{C})$.

Hence for sufficiently small $t \neq 0$ we obtain that $A(t) = \rho_t(S_1)$ and $B(t) = \rho_t(S_2)$ generate $M(n, \mathbf{C})$. By Burnside's matrix theorem, such a representation ρ_t is irreducible

To conclude the proof of Theorem 1.1, we will prove that all irreducible representations sufficiently close to ϱ_λ are non-metabelian. In order to do so, we will make use of the following result of H. Boden and S. Friedel [4, Theorem 1.2]: for every irreducible metabelian representation $\rho\colon \Gamma \to \mathrm{SL}(n)$ we have $\operatorname{tr} \rho(S_1) = 0$. Now, we have $\operatorname{tr} \varrho_\lambda(S_1) = \lambda^{-1}(\lambda^n + n - 1)$ and we claim that $\lambda^n + n - 1 \neq 0$. Notice that $\alpha = \lambda^n$ is a root of the Alexander polynomial $\Delta_K(t)$ and $\lambda^n + n - 1 = 0$ would imply that $1 - n$ is a root of $\Delta_K(t)$. This would imply that $t + n - 1$ divides $\Delta_K(t)$ and hence n divides $\Delta_K(1) = \pm 1$ which is impossible since $n \geq 2$. Therefore, $\operatorname{tr}(\rho(S_1)) \neq 0$ for all irreducible representations sufficiently close to ϱ_λ. This proves Theorem 1.1. □

5.3 Remark Let $\rho_\lambda\colon \Gamma \to \mathrm{SL}(n)$ be the diagonal representation given by $\rho_\lambda(\mu) = \mathrm{diag}(\lambda^{n-1}, \lambda^{-1} I_{n-1})$ where μ is a meridian of K. The orbit $\mathcal{O}(\rho_\lambda)$ of ρ_λ under the action of conjugation of $\mathrm{SL}(n)$ is contained in the closure $\overline{\mathcal{O}(\varrho_\lambda)}$. Hence ϱ_λ and ρ_λ project to the same point χ_λ of the variety of characters $X_n(\Gamma) = R_n(\Gamma) /\!\!/ \mathrm{SL}(n)$.

It would be natural to study the local picture of the variety of characters $X_n(\Gamma) = R_n(\Gamma) /\!\!/ \mathrm{SL}(n)$ at χ_λ as done in [11, § 8]. Unfortunately, there are much more technical difficulties since in this case the quadratic cone $Q(\rho_\lambda)$ coincides with the Zariski tangent space $Z^1(\Gamma; \mathfrak{sl}(n)_{\rho_\lambda})$. Therefore the third obstruction has to be considered.

Proof of lemma 5.2. The proof follows exactly the proof of Proposition 5.4 in [3]. We denote by $\mathcal{A}_t \subset \mathfrak{gl}(n)$ the algebra generated by $A(t)$ and $B(t)$.

For any matrix A we let $P_A(X)$ denote its characteristic polynomial. We have $P_{A_{11}(0)} = (\lambda^{-1} - X)^{n-1}$ and $a_{11}(0) = \lambda^{n-1}$. Since $\alpha = \lambda^n \neq 1$ we obtain $P_{A_{11}(0)}(a_{11}(0)) \neq 0$. It follows that $P_{A_{11}(t)}(a_{11}(t)) \neq 0$ for small t and hence

$$\frac{1}{P_{A_{11}(t)}(a_{11}(t))} P_{A_{11}(t)}(A(t)) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes (1, 0, \ldots, 0) \in \mathbf{C}[A(t)] \subset \mathcal{A}_t \,.$$

In the next step we will prove that

$$\mathcal{A}_t \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{C}^n \text{ and } (1, 0, \ldots, 0)\mathcal{A}_t = \mathbf{C}^n \,, \text{ for small } t \in \mathbf{C}^n \,.$$

It follows from this that $\mathcal{A}_t$ contains all rank one matrices since a rank one matrix can be written as $v \otimes w$ where v is a column vector and w is a row vector. Note also that $A(v \otimes w) = (Av) \otimes w$ and $(v \otimes w)A = v \otimes (wA)$. Since each matrix is the sum of rank one matrices the proposition follows.

Now consider the vectors

$$(1, 0, \ldots, 0)A(0),\ (1, 0, \ldots, 0)B(0), \ldots, (1, 0, \ldots, 0)B(0)^{n-1}.$$

Then for $1 \leq k \leq n-1$:

$$(1, 0, \ldots, 0)B(0)^k = \lambda^{-k}(\alpha^k, b\sum_{j=0}^{k-1} \alpha^{k-1-j} J^j)$$

and the dimension D of the vector space

$$\langle (1, 0, \ldots, 0)A(0), (1, 0, \ldots, 0)B(0), \ldots (1, 0, \ldots, 0)B(0)^{n-1} \rangle$$

is equal to

$$\begin{aligned} D &= \dim\langle (\alpha, 0), (\alpha, b), (\alpha^2, \alpha b + bJ), \ldots, (\alpha^{n-1}, b\sum_{j=0}^{k-1} \alpha^{k-1-j} J^j) \rangle \\ &= \dim\langle (\alpha, 0), (0, b), (0, bJ), \cdots (0, bJ^{n-2}) \rangle. \end{aligned}$$

Here, $J = J_{n-1} = I_{n-1} + N_{n-1}$ where $N_{n-1} \in GL(n-1, \mathbf{C})$ is the upper triangular Jordan normal form of a nilpotent matrix of degree $n-1$. Then a direct calculation gives that

$$\dim\langle b, bJ, \ldots, bJ^{n-2} \rangle = \dim\langle b, bN, \ldots, bN^{n-2} \rangle = n - 1 \,, \text{ as } b_1 \neq 0.$$

Thus $\dim\langle(1,0,\dots,0)A(0),(1,0,\dots,0)B(0),\dots(1,0,\dots,0)B(0)^{n-1}\rangle = n$ and the vectors

$$(1,0,\dots,0)A(0),\ (1,0,\dots,0)B(0),\dots,(1,0,\dots,0)B(0)^{n-1}$$

form a basis of the space of row vectors. This proves that $(1,0,\dots,0)\mathcal{A}_t$ is the space of row vectors for sufficiently small t.

In the final step consider the n column vectors

$$a_1(t) = A(t)\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\ a_i(t) = A^i(t)B(t)\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\ 0 \le i \le n-2$$

and write $B(t)\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix} b_{11}(t)\\ \mathbf{b}(t)\end{pmatrix}$ for the first column of $B(t)$; then

$$a_1(t) = \begin{pmatrix} a_{11}(t)\\ \mathbf{0}\end{pmatrix},\ a_{i+2}(t) = A^i(t)\begin{pmatrix} b_{11}(t)\\ \mathbf{b}(t)\end{pmatrix},\ 0 \le i \le n-2.$$

Define the function $f(t) := \det(a_1(t),\dots,a_n(t))$ and $g(t)$ by:

$$f(t) = a_{11}(t)g(t), \text{ where } g(t) = \det\left(\mathbf{b}(t), A_{11}(t)\mathbf{b}(t),\dots,A_{11}^{n-2}(t)\mathbf{b}(t)\right).$$

Now, for $k \ge 0$ the k-th derivative $g^{(k)}(t)$ of $g(t)$ is given by:

$$\sum_{s_1,\dots,s_{n-1}} c_{s_1,\dots,s_{n-1}} \det\left(\mathbf{b}^{(s_1)}(t), (A_{11}(t)\mathbf{b}(t))^{(s_2)},\dots,(A_{11}^{n-2}(t)\mathbf{b}(t))^{(s_{n-1})}\right)$$

where

$$c_{s_1,\dots,s_{n-1}} = \begin{cases} \binom{k}{s_1,\dots,s_{n-1}} = \frac{k!}{s_1!\dots s_{n-1}!} & \text{if } s_1+\dots+s_{n-1} = k;\\ 0 & \text{othewise.}\end{cases}$$

As $\mathbf{b}(0) = 0$ one have, for $0 \le k \le n-2$, $g^{(k)}(0) = 0$ and consequently $f^{(k)}(0) = 0$ for all $0 \le k \le n-2$.
Now, for $k = n-1$, we have

$$\begin{aligned}\frac{g^{(n-1)}(0)}{(n-1)!} &= \det\left(\mathbf{b}'(0), (A_{11}(t)\mathbf{b}(t))'(0),\dots,(A_{11}^{n-2}(t)\mathbf{b}(t))'(0)\right)\\ &= \det\left(\mathbf{b}'(0), A_{11}(0)\mathbf{b}'(0),\dots,A_{11}^{n-2}(0)\mathbf{b}'(0)\right)\\ &= \det\left(\mathbf{b}'(0), (\lambda^{-1}J)\mathbf{b}'(0),\dots,(\lambda^{-1}J)^{n-2}\mathbf{b}'(0)\right)\\ &= \det\left(\mathbf{b}'(0), \lambda^{-1}N\mathbf{b}'(0),\dots,\lambda^{-(n-2)}N^{n-2}\mathbf{b}'(0)\right)\\ &\ne 0 \text{ since } b'_{n1} \ne 0.\end{aligned}$$

Thus, $f^{(n-1)}(0) = a_{11}(0)g^{(n-1)}(0) \neq 0$ and $f(t) \neq 0$ for sufficiently small t, $t \neq 0$. $\square$

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